All-Pairs Shortest Paths - Floyd's Algorithm

Parallel and Distributed Computing

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Outline

- All-Pairs Shortest Paths, Floyd's Algorithm
    - Partitioning
    - Input / Output
    - Implementation and Analysis
    - Benchmarking

Shortest Paths

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Given a weighted, directed graph $G(V, E)$, determine the shortest path between any two nodes in the graph.

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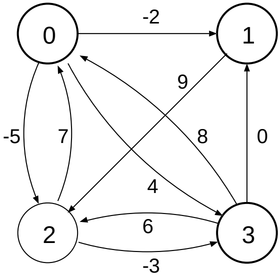

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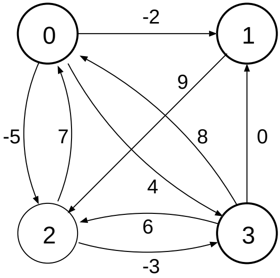

$$\begin{bmatrix} 0 & -2 & -5 & 4 \\ \infty & 0 & 9 & \infty \\ 7 & \infty & 0 & -3 \\ 8 & 0 & 6 & 0 \end{bmatrix}$$

Adjacency Matrix

The Floyd-Warshall Algorithm

Recursive solution based on *intermediate* vertices.

Let p_{ij} be the minimum-weight path from node i to node j among paths that use a subset of intermediate vertices $\{0, \ldots, k-1\}$.

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Consider an additional node k:

$k \notin p_{ij}$

then p_{ij} is shortest path considering the subset of intermediate vertices $\{0, \ldots, k\}$.

$k \in p_{ij}$

then we can decompose p_{ij} as $i \overset{p_{ik}}{\rightsquigarrow} k \overset{p_{kj}}{\rightsquigarrow} j$, where subpaths p_{ik} and p_{kj} have intermediate vertices in the set $\{0, \ldots, k-1\}$.

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$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = -1 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 0 \end{cases}$$

The Floyd-Warshall Algorithm

1. for $k \leftarrow 0$ to $|V| - 1$
2.   for $i \leftarrow 0$ to $|V| - 1$
3.   for $j \leftarrow 0$ to $|V| - 1$
4. $d[i,j] \leftarrow \min(d[i,j], d[i,k] + d[k,j])$

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Complexity?

The Floyd-Warshall Algorithm

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Complexity: $\Theta(|V|^3)$

Partitioning

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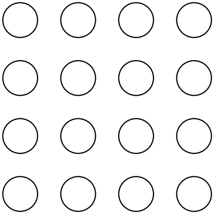
Domain decomposition: divide adjacency matrix into its $|V|^2$ elements (computation in the inner loop is primitive task).



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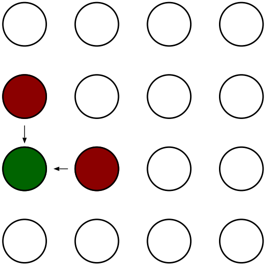
Let $k = 1$. Row sweep, $i = 2$.



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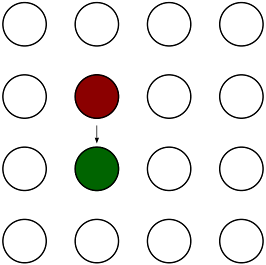

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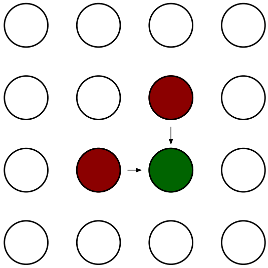

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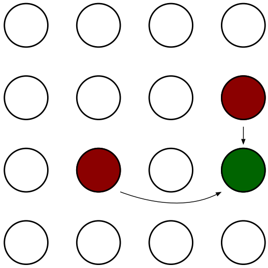

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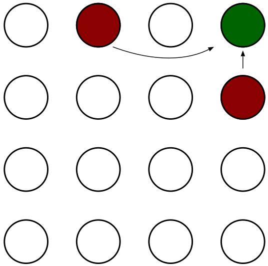

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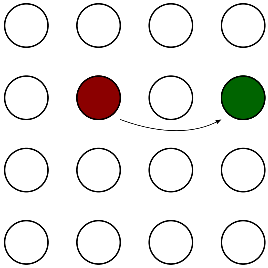

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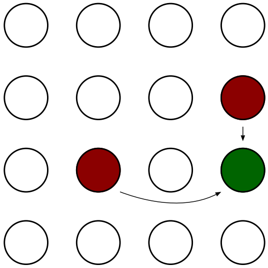

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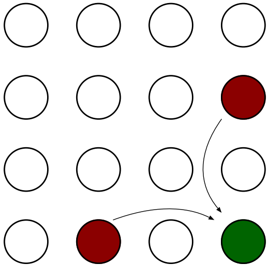
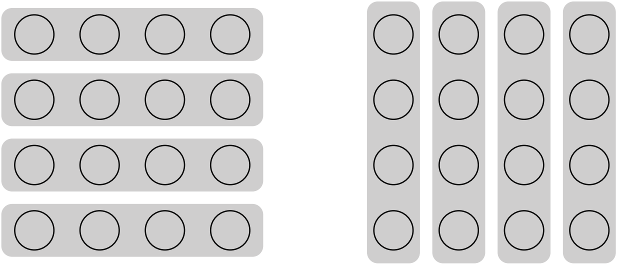


Communication:

In iteration k, every task in row/column k broadcasts its value within task row/column.



Agglomeration and Mapping

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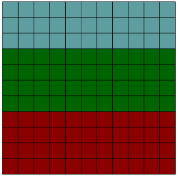
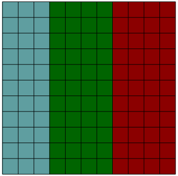
- create one task per MPI process
- agglomerate tasks to minimize communication

Possible decompositions: row-wise vs column-wise block striped $(n = 11, p = 3)$.





Relative merit?

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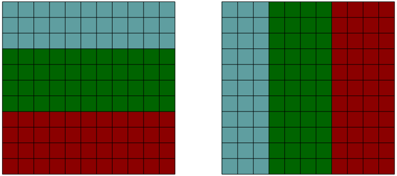


Relative merit?

- Column-wise block striped
  - Broadcast within columns eliminated
- Row-wise block striped
  - Broadcast within rows eliminated
  - Reading, writing and printing matrix simpler

Comparing Decompositions

Choose row-wise block striped decomposition.

Some tasks get $\left\lceil \frac{n}{p} \right\rceil$ rows, other get $\left\lfloor \frac{n}{p} \right\rfloor$.

Which task gets which size?

Comparing Decompositions

Choose row-wise block striped decomposition.

Some tasks get $\left\lceil \frac{n}{p} \right\rceil$ rows, other get $\left\lfloor \frac{n}{p} \right\rfloor$.

Which task gets which size?

Distributed approach: distribute larger blocks evenly.

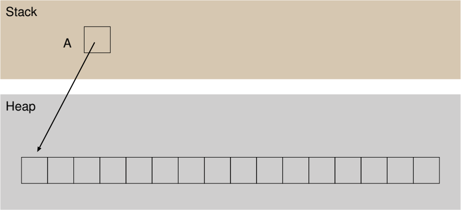
First element of task i: $\left\lfloor i\frac{n}{p} \right\rfloor$

Last element of task i: $\left\lfloor (i+1)\frac{n}{p} \right\rfloor - 1$

Task owner of element j: $\lfloor (p(j+1)-1)/n \rfloor$

Dynamic Matrix Allocation

Array allocation:



Matrix allocation:

Dynamic Matrix Allocation

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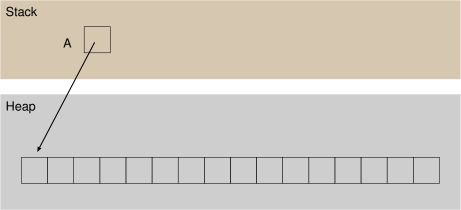

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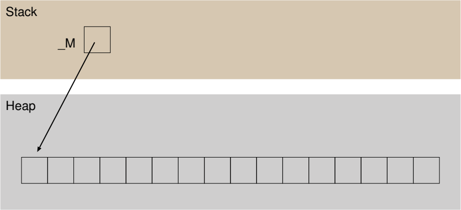

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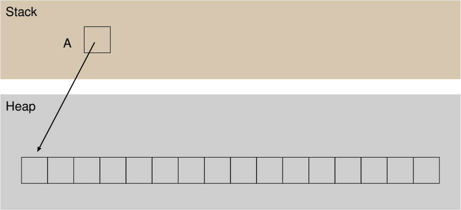

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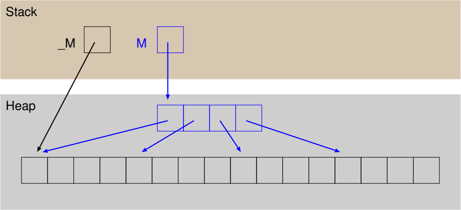
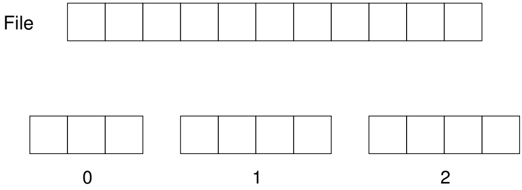


Reading the Graph Matrix



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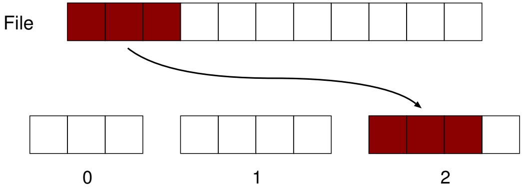

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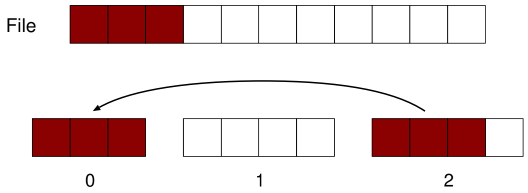

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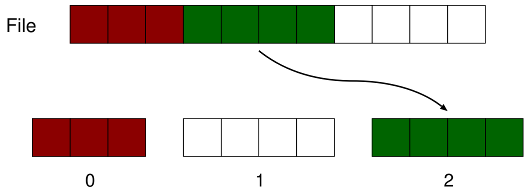

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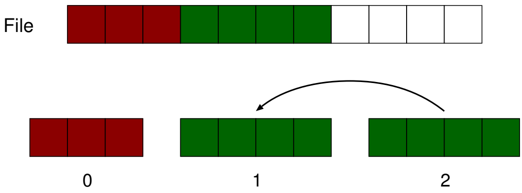

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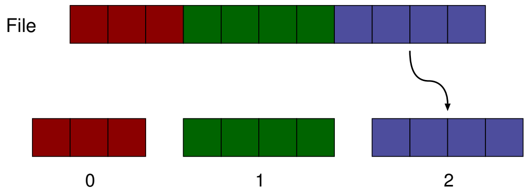

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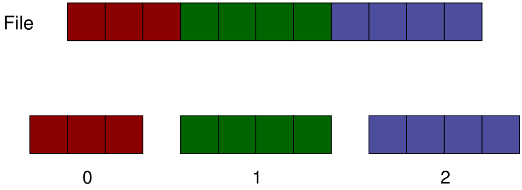

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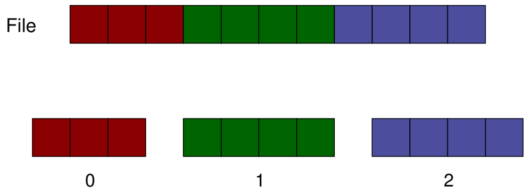
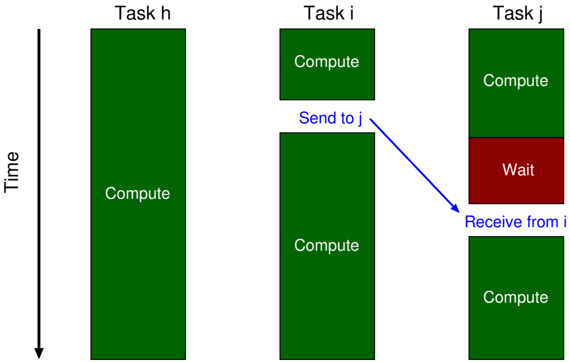


Why don't we read the whole file and then execute a MPI_Scatter?

Point-to-point Communication

- involves a pair of processes
  - one process sends a message
  - other process receives the message



MPI_Send

```
int MPI_Send (
      void          *message,
      int            count,
      MPI_Datatype   datatype,
      int            dest,
      int            tag,
      MPI_Comm       comm
)
```

MPI_Recv

```
int MPI_Recv (
      void          *message,
      int            count,
      MPI_Datatype   datatype,
      int            source,
      int            tag,
      MPI_Comm       comm,
      MPI_Status    *status
)
```

Coding Send / Receive

```
...
if (id == j) {
   ...
   Receive from i
   ...
}

...

if (id == i) {
   ...
   Send to j
   ...
}
...
```

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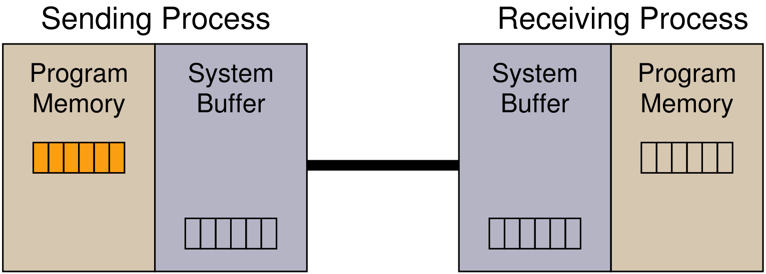
```

Receive is before Send! Why does this work?

Internals of Send and Receive



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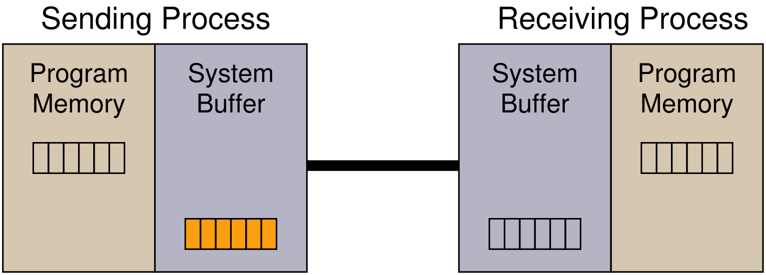

`MPI_Send`

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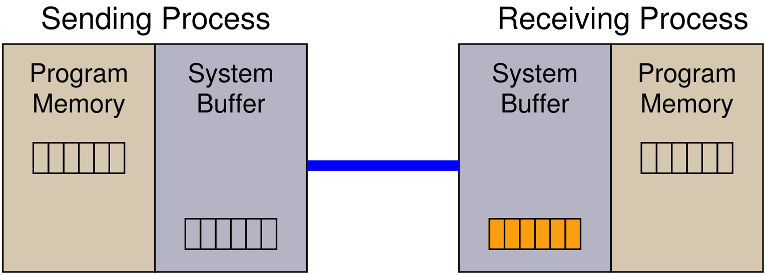

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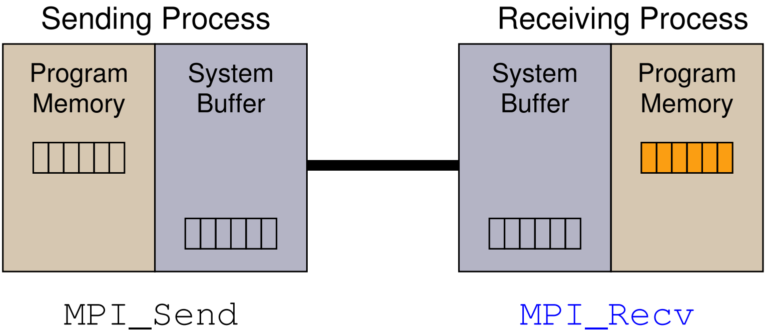

Return from MPI_Send

- function blocks until message buffer free

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- message buffer is free when
  - message copied to system buffer, or
  - message transmitted

Return from MPI_Send

- function blocks until message buffer free
- message buffer is free when
  - message copied to system buffer, or
  - message transmitted
- typical scenario
  - message copied to system buffer
  - transmission overlaps computation

Return from MPI_Recv

- function blocks until message in buffer

Return from MPI_Recv

- function blocks until message in buffer

- if message never arrives, function never returns!

Deadlock

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Process waiting for a condition that will never become true.

Easy to write send/receive code that deadlocks:

- two processes: both receive before send

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Deadlock

Deadlock

Process waiting for a condition that will never become true.

Easy to write send/receive code that deadlocks:

- two processes: both receive before send
- send tag doesn't match receive tag
- process sends message to wrong destination process

C Code

```
void compute_shortest_paths (int id, int p, double **a, int n)
{
   int  i, j, k;
   int  offset;   /* Local index of broadcast row */
   int  root;     /* Process controlling row to be bcast */
   double* tmp;      /* Holds the broadcast row */

   tmp = (double *) malloc (n * sizeof(double));
   for (k = 0; k < n; k++) {
      root = BLOCK_OWNER(k,p,n);
      if (root == id) {
         offset = k - BLOCK_LOW(id,p,n);
         for (j = 0; j < n; j++)
            tmp[j] = a[offset][j];
      }
      MPI_Bcast (tmp, n, MPI_DOUBLE, root, MPI_COMM_WORLD);
      for (i = 0; i < BLOCK_SIZE(id,p,n); i++)
         for (j = 0; j < n; j++)
            a[i][j] = MIN(a[i][j],a[i][k]+tmp[j]);
   }
   free (tmp);
}
```

Analysis of the Parallel Algorithm

Let α be the time to compute an iteration.
Sequential execution time?

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Sequential execution time: αn^3

Computation time of parallel program?

Analysis of the Parallel Algorithm

Let α be the time to compute an iteration.

Sequential execution time: αn^3

Computation time of parallel program: $\alpha n \left\lceil \frac{n}{p} \right\rceil n$

- innermost loop executed n times
- middle loop executed at most $\left\lceil \frac{n}{p} \right\rceil$ times
- outer loop executed n times

Number of broadcasts?

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Broadcast time: $\lceil \log p \rceil \left(\lambda + \frac{4n}{\beta} \right)$

- each broadcast has $\lceil \log p \rceil$ steps
- λ is the message latency
- β is the bandwidth
- each broadcast sends $4n$ bytes

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Expected parallel execution time: $\alpha n^2 \left\lceil \frac{n}{p} \right\rceil + n \lceil \log p \rceil \left(\lambda + \frac{4n}{\beta} \right)$

Analysis of the Parallel Algorithm

Previous expression will overestimate parallel execution time: after the first iteration, broadcast transmission time overlaps with computation of next row.

Expected parallel execution time:

$$\alpha n^2 \left\lceil \frac{n}{p} \right\rceil + n\lceil \log p \rceil \lambda + \lceil \log p \rceil \frac{4n}{\beta}$$

Experimental measurements:

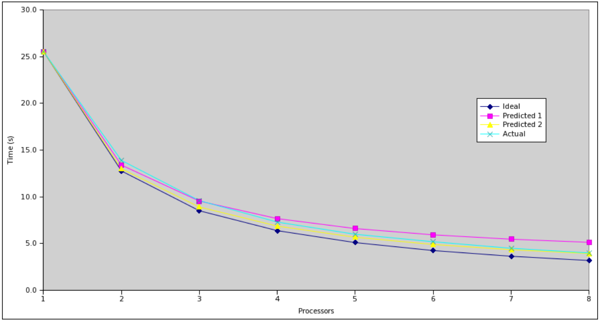
$\alpha = 25, 5$ ns

$\lambda = 250$ μs

$\beta = 10^7$ bytes/s

Experimental Results

Procs	Ideal	Predict 1	Predict 2	Actual
1	25,5	25,5	25,5	25,5
2	12,8	13,4	13,0	13,9
3	8,5	9,5	8,9	9,6
4	6,4	7,7	6,9	7,3
5	5,1	6,6	5,7	6,0
6	4,3	5,9	4,9	5,2
7	3,6	5,5	4,3	4,5
8	3,2	5,1	3,9	4,0



Review

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  - Implementation and Analysis
  - Benchmarking

Next Class

- Performance metrics